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A remark on weakly convex continuous mappings in topological linear spaces

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ABSTRACT

Let C be a compact convex subset of a Hausdorff topological linear space and $T: C \to C$ a continuous mapping. We characterize those mappings T for which $T(C)$ is convexly totally bounded.

1. Introduction

In this note X denotes a Hausdorff topological linear space. A subset K of X is said to be *convexly totally bounded* (briefly *ctb*) if for every 0-neighborhood U there are points $x_1, \ldots, x_n \in K$ and convex subsets $C_1, \ldots, C_n$ of U such that $K \subset \bigcup_{i=1}^{n}(x_i + C_i)$. For example, every compact set in a locally convex space is ctb. This definition has been introduced by Idzik [4] in connection with the well-known Schauder conjecture [6, Problem 54]: *Every compact convex set in a Hausdorff topological linear space has the fixed point property.* Then, Idzik has proved that the conjecture holds if C is ctb.

Theorem 1. *([4, Theorem 2.4]) Let C be a compact convex subset of X and $T: C \to C$ be a continuous mapping. If $T(C)$ is ctb, then T has fixed point in C.*

We recall that examples of compact convex sets in Hausdorff topological linear spaces which are not ctb have been provided in [2,8,9]. The constructions of these sets are based on Roberts' example (see [7,5]) for a compact convex set without extreme points, and each set has the fixed point property.

Aim of this note is to characterize, among continuous mappings, those mappings T for which $T(C)$ is a ctb set. Moreover, we characterize ctb subsets of X in terms of a weakened Zima type property.

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2. Main result

In the following without loss of generality we may choose a fundamental system of 0-neighborhoods consisting of closed, symmetric and circled sets. Moreover, we assume that every 0-neighborhood we consider belongs to such a system. For a subset A of X, we denote the convex hull of A by $\operatorname{co} A$. Given a continuous mapping $T : X \to X$, we strength the continuity of T by introducing the following notion.

Definition 2. A mapping $T : X \to X$ is said to be *weakly convex continuous* at $x_0 \in X$ if, for every 0-neighborhood N, there exist a 0-neighborhood V and subsets $A_1, \dots, A_n$ of X such that

(i) $T(x_0 + V) = \bigcup_{i=1}^{n} A_i$;
(ii) $\operatorname{co} A_i \subset T(x_0) + N$, for $i = 1, \dots, n$.

Clearly, in a locally convex space, the class of weakly convex continuous mappings coincides with that of continuous one. Moreover, if C is a compact convex set in X which is not ctb, then the identity mapping on C is continuous but not weakly convex continuous.

Theorem 3. *Let C be a compact convex subset of X and $T : C \to C$ be a continuous mapping. Then T is weakly convex continuous in C if and only if $T(C)$ is ctb.*

Proof. Assume that T is weakly convex continuous in C. Let N be a 0-neighborhood and $x \in C$. Find a 0-neighborhood V_x and subsets $A_1^x, \dots, A_{k_x}^x$ of X such that $T(x + V_x) = \bigcup_{j=1}^{k_x} A_j^x$ and $\operatorname{co} A_j^x \subset T(x) + N$, for $j = 1, \dots, k_x$. Then $\{x + V_x\colon x \in C\}$ is an open covering of C, hence there is a finite subcovering $\{x_1 + V_{x_1}, \dots, x_n + V_{x_n}\}$ of C. Therefore

$$T(C) = \bigcup_{i=1}^{n} T(x_i + V_{x_i}) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{k_{x_i}} A_j^{x_i} \subset \bigcup_{i=1}^{n} \bigcup_{j=1}^{k_{x_i}} \operatorname{co} A_j^{x_i}.$$

Set $C_j^i = \operatorname{co} A_j^{x_i} - T(x_i)$, for $i = 1, \dots, n$ and $j = 1, \dots, k_{x_i}$. Since $\operatorname{co} A_j^{x_i} \subset T(x_i) + N$, then C_j^i are convex subsets of N and

$$T(C) \subset \bigcup_{i=1}^{n} \bigcup_{j=1}^{k_{x_i}} \bigl(T(x_i) + C_j^i\bigr).$$

Hence $T(C)$ is ctb.

To prove the converse, assume that $T(C)$ is a ctb set. Let $x \in C$ and N be a 0-neighborhood. Choose a 0-neighborhood U such that $U + U \subset N$. By the continuity of T there is a 0-neighborhood V such that

$$T(x + V) \subset T(x) + U.$$

Since $T(C)$ is ctb, then $T(x + V)$ is ctb. Hence there are points $x_1, \dots, x_n \in x + V$ and convex subsets $C_1, \dots, C_n$ of U such that $T(x + V) \subset \bigcup_{i=1}^{n} (T(x_i) + C_i)$. Now for all $z \in T(x_i) + C_i$, we can write

$$z = z - T(x_i) + T(x_i) \in C_i + T(x + V),$$

so that $z \in U + U + T(x) \subset T(x) + N$, therefore

$$T(x_i) + C_i \subset T(x) + N.$$

Choosing $A_i = (T(x_i) + C_i) \cap T(x + V)$, for $i = 1, \dots, n$, we have that $T(x + V) = \bigcup_{i=1}^{n} A_i$ and $\operatorname{co} A_i \subset T(x) + N$, for $i = 1, \dots, n$. This completes the proof. $\square$

Corollary 4. *Let C be a compact convex subset of X and $T : C \to C$ a weakly convex continuous mapping. Then T has fixed point in C.*

Remark 5. In [1] a mapping $T : C \to C$ is called *convex continuous* at $x_0 \in C$ if, for every 0-neighborhood N, there is a 0-neighborhood V such that $\operatorname{co} T(x_0 + V) \subset T(x_0) + N$.

Clearly, T convex continuous implies T weakly convex continuous. Then the result on the existence of fixed points of convex continuous mappings [1, Theorem 2.1] yields as a corollary of Idzik Theorem, without any additional hypothesis on the space X, as required in [1].

Remark 6. A subset K of X is said to be *strongly convexly totally bounded* (briefly *sctb*) (see [2, Definition 2.1]) if for every 0-neighborhood U there are points $x_1, \dots, x_n \in X$ and a convex subset C_0 of U such that $K \subset \bigcup_{i=1}^n (x_i + C_0)$. Then given $T : C \to C$, following the proof of the sufficient part of Theorem 3, we get that T is convex continuous if $T(C)$ is sctb. Assuming that C is compact convex and ctb, then the problem if every weakly convex continuous mapping is convex continuous is equivalent to the unsolved problem whether every compact convex ctb set is sctb (see [2,9]).

Now we give the following definition which is a modification of the Zima type property (see [10] and [3]).

Definition 7. A subset K of X is said to be of the *weak Zima type* if for every 0-neighborhood U there are subsets $K_1, \dots, K_n$ of X and a 0-neighborhood V such that

$$K = \bigcup_{i=1}^n K_i \quad \text{and} \quad \operatorname{co}\big(V \cap (K_i - K_i)\big) \subset U \quad \text{for } i = 1, \dots, n.$$

The following proposition characterizes ctb subsets of X.

Proposition 8. *Let K be a totally bounded subset of X. Then K is ctb if and only if K is of the weak Zima type.*

Proof. Let K be a ctb subset of X. Let U and W be 0-neighborhood such that $W + W \subset U$. Find points $x_1, \dots, x_n \in K$ and convex subsets $C_1, \dots, C_n$ of W such that $K \subset \bigcup_{i=1}^n (x_i + C_i)$. Put $K_i = K \cap (x_i + C_i)$ for each i, then

$$K = \bigcup_{i=1}^n K_i \quad \text{and} \quad K_i - K_i \subset C_i - C_i \subset W + W \subset U.$$

Hence $\operatorname{co}(V \cap (K_i - K_i)) \subset U$, for every 0-neighborhood V and for all i.

Vice versa, let K be a totally bounded subset of X of the weak Zima type and U a 0-neighborhood. Find subsets $K_1, \dots, K_n$ of X and a 0-neighborhood V such that

$$K = \bigcup_{i=1}^n K_i \quad \text{and} \quad \operatorname{co}\big(V \cap (K_i - K_i)\big) \subset U.$$

For each $i = 1, \dots, n$, the sets K_i are totally bounded, hence there exist $x_1^i, \dots, x_{m_i}^i \in K_i$ such that

$$K_i \subset \bigcup_{j=1}^{m_i} \big(x_j^i + V\big).$$

Fixed i, for all $j = 1, \dots, m_i$ we have that

$$\big(K_i \cap \big(x_j^i + V\big)\big) - x_j^i \subset V \quad \text{and} \quad \big(K_i \cap \big(x_j^i + V\big)\big) - x_j^i \subset K_i - x_j^i.$$

Hence

$$\big(K_i \cap \big(x_j^i + V\big)\big) - x_j^i \subset V \cap \big(K_i - x_j^i\big) \subset V \cap (K_i - K_i) \subset \operatorname{co}\big(V \cap (K_i - K_i)\big) \subset U.$$

Setting $C_i = \operatorname{co}(V \cap (K_i - K_i))$ we have $K \subset \bigcup_{i=1}^n \bigcup_{j=1}^{m_i} (x_j^i + C_i)$. This completes the proof. □

In [9] Weber has proved that if K is totally bounded, then K is sctb if and only if K is of the Zima type.

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